

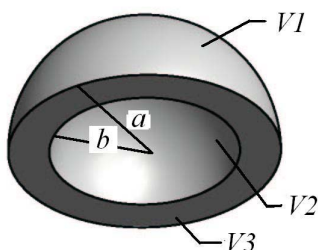
記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Useful formulas

$$\nabla V = \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} \quad \text{and} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi$$

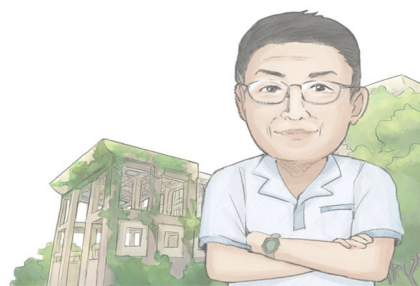
1. (6%, 7%, 7%) Suppose the potential at the surface of a hollow hemisphere is specified, as shown in the figure, where $V_1(a, \theta) = 0$, $V_2(b, \theta) = V_0(2 \cos \theta - 5 \cos \theta \sin^2 \theta)$, $V_3(r, \pi/2) = 0$. V_0 is a constant.
 - (a) Show the general solution in the region $b \leq r \leq a$.
 - (b) Determine the potential in the region $b \leq r \leq a$, using the boundary conditions.
 - (c) Calculate the electric field in the region $b \leq r \leq a$.

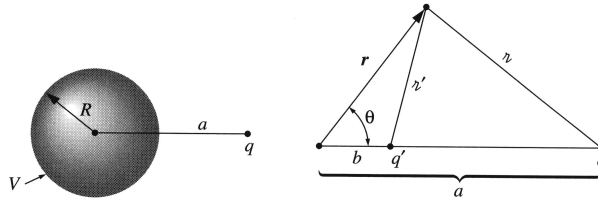
[Hint: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, and $P_3(x) = (5x^3 - 3x)/2$.]



2. (10%, 10%) A point charge q is situated a large distance \mathbf{r} from a neutral atom of polarizability α .
 - (a) Find the induced dipole moment of the atom \mathbf{p} .
 - (b) Find the force between them (attractive or repulsive).
3. (10%, 10%) A point charge q is situated at distance a from the center of a conducting sphere of radius R . The sphere is maintained at the constant potential V_0 .
 - (a) Find the position and the value of the image charge.
 - (b) Verify that the tangential component of the electric field is zero throughout on the surface of the metal sphere.

[Hint: 1. use the notations shown below. 2. Assume q lays on the z -axis]





4. (8%, 6%, 6%) Consider two infinite parallel metal plates separated by a distance s are at potential 0 and V_0 as shown in the figure. .

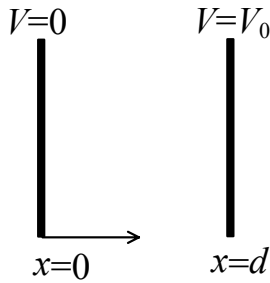
(a) Use Poisson's equation to find the potential V in the region between the plates where the

space charge density is $\rho = \rho_0 \frac{x}{d}$.

(b) Find the electric field \mathbf{E} in the region between the plates.

(c) Use the boundary condition to determine the charge densities on the plates.

[Note: The electric field inside the metal plate is zero $E_{inside} = 0$.]



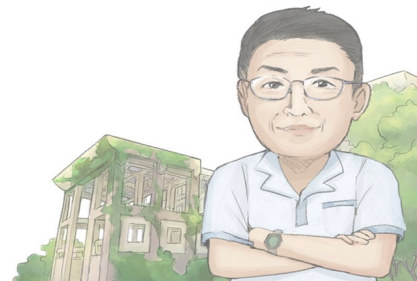
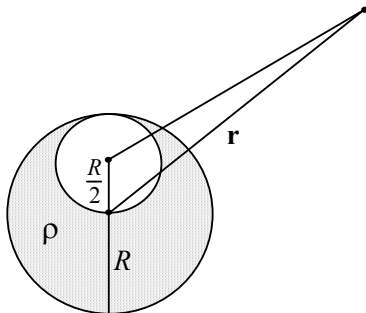
5. (8%, 6%, 6%) Consider a hollowed charged sphere with radius R and uniform charge density ρ as shown in the figure. The inner radius of the spherical cavity is $R/2$.

(a) If the observer is very far from the charged sphere, find the multiple expansion of the potential V in power of $1/r$

(b) Find the dipole moment \mathbf{p} .

(c) Find the electric field \mathbf{E} up to the dipole term.

[Note: Specify a vector with both magnitude and direction.]



1.

(a)

$$\text{Boundary condition} \begin{cases} \text{(i)} V_1(a, \theta) = 0 \\ \text{(ii)} V_2(b, \theta) = V_0(2 \cos \theta - 5 \cos \theta \sin^2 \theta) = V_0(5 \cos^3 \theta - 3 \cos \theta) = 2V_0 P_3 \\ \text{(iii)} V_3(r, \theta = \pi/2) = 0 \end{cases}$$

$$\text{General solution } V(r, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-(\ell+1)}) P_{\ell}(\cos \theta)$$

(b)

$$\text{B.C. (i)} \rightarrow V(a, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} a^{\ell} + B_{\ell} a^{-(\ell+1)}) P_{\ell}(\cos \theta) = 0 \Rightarrow B_{\ell} = -A_{\ell} a^{2\ell+1}$$

$$\text{B.C. (ii)} \rightarrow V(b, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} b^{\ell} + B_{\ell} b^{-(\ell+1)}) P_{\ell}(\cos \theta) = 2V_0 P_3(\cos \theta)$$

Comparing the coefficient $\Rightarrow A_3 b^3 + B_3 b^{-4} = 2V_0, A_{\ell} = B_{\ell} = 0$ for $\ell = 0, 1, 2, 4, 5, \dots$

$$\text{B.C. (iii)} \rightarrow V(r, \theta = \frac{\pi}{2}) = (A_3 r^3 + B_3 r^{-4}) P_3(0) = 0$$

$\Rightarrow A_{\ell} = B_{\ell} = 0$ except $\ell = 3$,

$$A_3 = \frac{2V_0 b^4}{b^7 - a^7} \text{ and } B_3 = -\frac{2V_0 b^4 a^7}{b^7 - a^7}$$

$$\therefore V(r, \theta) = \left(\frac{2V_0}{b^7 - a^7} b^4 r^3 - \frac{2V_0}{b^7 - a^7} b^4 a^7 r^{-4} \right) \left(\frac{5 \cos^3 \theta - 3 \cos \theta}{2} \right)$$

(c)

$$V(r, \theta) = \left(\frac{2V_0}{b^7 - a^7} b^4 r^3 - \frac{2V_0}{b^7 - a^7} b^4 a^7 r^{-4} \right) \left(\frac{5 \cos^3 \theta - 3 \cos \theta}{2} \right)$$

$$\begin{aligned} \mathbf{E} &= -\nabla V = -\frac{\partial V}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} \\ &= -\left(\frac{6V_0}{b^7 - a^7} b^4 r^2 - \frac{8V_0}{b^7 - a^7} b^4 a^7 r^{-5} \right) \left(\frac{5 \cos^3 \theta - 3 \cos \theta}{2} \right) \hat{\mathbf{r}} \\ &\quad + \left(\frac{6V_0}{b^7 - a^7} b^4 r^2 - \frac{8V_0}{b^7 - a^7} b^4 a^7 r^{-5} \right) \left(\frac{15 \cos^2 \theta \sin \theta - 3 \sin \theta}{2} \right) \hat{\boldsymbol{\theta}} \end{aligned}$$

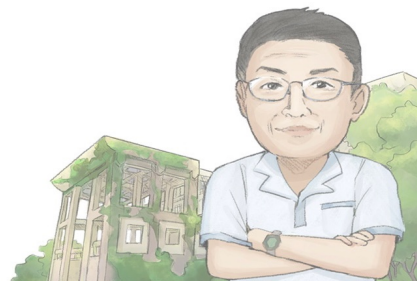
2.

(a)

$$\text{The electric field of a point charge } \mathbf{E} = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \hat{\mathbf{r}}$$

$$\text{The induced dipole moment } \mathbf{p} = \alpha \mathbf{E} = \frac{1}{4\pi\epsilon} \frac{\alpha q}{r^2} \hat{\mathbf{r}}$$

(b)



The total electric static energy $U = -(\frac{1}{2})\mathbf{p} \cdot \mathbf{E} = -(\frac{1}{2})(\frac{1}{4\pi\epsilon})^2 \frac{\alpha q^2}{r^4}$

Note: $(\frac{1}{2})$ comes from the fact that the dipole moment \mathbf{p} is induced by \mathbf{E} .

The force between is $\mathbf{F} = -\nabla U = -2\alpha(\frac{1}{4\pi\epsilon})^2 \frac{q^2}{r^5} \hat{\mathbf{r}}$ attractive.

The direction of the induced dipole \mathbf{p} is in line with the electric field \mathbf{E} generated by the charge q .

3.(a)

Assume the image charge q' is placed at a distance b from the center of the sphere.

It is equipotential on the surface of a grounded sphere.

Using two boundary conditions at P_1 and P_2

$$\left. \begin{aligned} \text{At } P_1: \frac{1}{4\pi\epsilon_0} \left(\frac{q'}{R-b} + \frac{q}{a-R} \right) &= 0 \\ \text{At } P_2: \frac{1}{4\pi\epsilon_0} \left(\frac{q'}{R+b} + \frac{q}{a+R} \right) &= 0 \end{aligned} \right\} \text{two equations and two unknowns } (q' \text{ and } b)$$

$$b = \frac{R^2}{a} \text{ (position), } q' = -\frac{R}{a} q \text{ (value of the image charge)}$$

(b)

The potential outside the sphere when $V=0$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q'}{|\mathbf{r} - b\hat{\mathbf{z}}|} + \frac{q}{|\mathbf{r} - a\hat{\mathbf{z}}|} \right\}, \text{ where } b = \frac{R^2}{a} \text{ and } q' = -\frac{R}{a} q$$

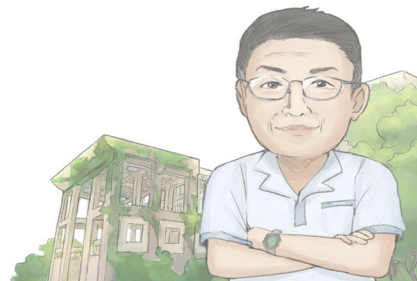
The potential outside the sphere when $V=V_0$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{4\pi\epsilon_0 R V_0}{r} + \frac{q'}{|\mathbf{r} - b\hat{\mathbf{z}}|} + \frac{q}{|\mathbf{r} - a\hat{\mathbf{z}}|} \right\}$$

where $\begin{cases} |\mathbf{r} - b\hat{\mathbf{z}}| = \sqrt{(r^2 \sin^2 \theta + (r \cos \theta - b)^2)} = \sqrt{(r^2 - 2br \cos \theta + b^2)} \\ |\mathbf{r} - a\hat{\mathbf{z}}| = \sqrt{(r^2 \sin^2 \theta + (r \cos \theta - a)^2)} = \sqrt{(r^2 - 2ar \cos \theta + a^2)} \end{cases}$

On the surface of the metal sphere, $\mathbf{E} = -\nabla V(\mathbf{r}) = -\frac{\partial V}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}}$

$$\begin{aligned} E_\theta &= -\frac{1}{r} \frac{\partial V}{\partial \theta} = -\frac{1}{4\pi\epsilon_0} \frac{1}{r} \frac{\partial}{\partial \theta} \left\{ \frac{q'}{\sqrt{(r^2 - 2br \cos \theta + b^2)}} + \frac{q}{\sqrt{(r^2 - 2ar \cos \theta + a^2)}} \right\} \\ &= -\frac{1}{4\pi\epsilon_0} \frac{1}{r} \left(\frac{q'(-2br \sin \theta)}{(r^2 - 2br \cos \theta + b^2)^{3/2}} + \frac{q(-2ar \sin \theta)}{(r^2 - 2ar \cos \theta + a^2)^{3/2}} \right) \\ &= \frac{2 \sin \theta}{4\pi\epsilon_0} \left(\frac{q'b}{(r^2 - 2br \cos \theta + b^2)^{3/2}} + \frac{qa}{(r^2 - 2ar \cos \theta + a^2)^{3/2}} \right) \end{aligned}$$



$$\frac{q'b}{(R^2 + 2bR \cos \theta - b^2)^{3/2}} = \frac{-\frac{R}{a}q \frac{R^2}{a}}{(R^2 - 2\frac{R^2}{a}R \cos \theta + \frac{R^4}{a^2})^{3/2}} = \frac{-\frac{R^3}{a^2}q}{\frac{R^3}{a^3}(a^2 - 2aR \cos \theta + R^2)^{3/2}}$$

$$= -\frac{qa}{(R^2 - 2aR \cos \theta + a^2)^{3/2}}$$

$$E_\theta(@r=R) = \frac{2 \sin \theta}{4\pi\epsilon_0} \left(-\frac{qa}{(R^2 - 2aR \cos \theta + a^2)^{3/2}} + \frac{qa}{(R^2 - 2aR \cos \theta + a^2)^{3/2}} \right) = 0$$

4.(a)

$$\frac{d^2V}{dx^2} = -\frac{\rho_0 x}{\epsilon_0 d} \Rightarrow V(x) = -\frac{\rho_0 x^3}{6\epsilon_0 d} + c_1 x + c_2$$

Use the boundary conditions to determine the coefficients.

$$\begin{cases} V(0) = 0 \Rightarrow c_2 = 0 \\ V(d) = V_0 = -\frac{\rho_0 d^2}{6\epsilon_0} + c_1 d \Rightarrow c_1 = \frac{V_0}{d} + \frac{\rho_0 d}{6\epsilon_0} \end{cases}$$

$$V(x) = -\frac{\rho_0 x^3}{6\epsilon_0 d} + \left(\frac{V_0}{d} + \frac{\rho_0 d}{6\epsilon_0} \right) x$$

(b)

$$V(x) = -\frac{\rho_0 x^3}{6\epsilon_0 d} + \left(\frac{V_0}{d} + \frac{\rho_0 d}{6\epsilon_0} \right) x$$

$$\mathbf{E} = -\nabla V(x) = \frac{\rho_0 x^2}{2\epsilon_0 d} - \left(\frac{V_0}{d} + \frac{\rho_0 d}{6\epsilon_0} \right) \hat{\mathbf{x}}$$

(c)

$$\begin{cases} \mathbf{E}_{x=0} = -\left(\frac{V_0}{d} + \frac{\rho_0 d}{6\epsilon_0} \right) \hat{\mathbf{x}} \\ \mathbf{E}_{x=d} = \left(\frac{2\rho_0 d}{3\epsilon_0} - \frac{V_0}{d} \right) \hat{\mathbf{x}} \end{cases} \Rightarrow \begin{cases} \sigma_{x=0} = \epsilon_0 (E_{\text{outside}} - E_{\text{inside}}) = -\left(\frac{\epsilon_0 V_0}{d} + \frac{\rho_0 d}{6} \right) \\ \sigma_{x=d} = \epsilon_0 (E_{\text{outside}} - E_{\text{inside}}) = \left(\frac{2\rho_0 d}{3} - \frac{\epsilon_0 V_0}{d} \right) \end{cases}$$

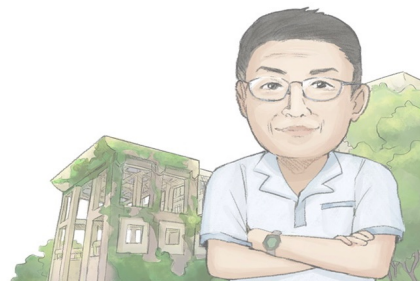
5.

(a)

Consider this problem as two charge spheres, one with charge density ρ the other with opposite charge density $-\rho$.

$$V_{\text{big}} = \frac{1}{4\pi\epsilon_0 r} \left(\rho \frac{4\pi}{3} R^3 \right) \quad \text{and} \quad V_{\text{small}} = \frac{1}{4\pi\epsilon_0 \left| \mathbf{r} - \frac{1}{2} \mathbf{R} \right|} \left(-\rho \frac{4\pi}{3} \left(\frac{R}{2} \right)^3 \right)$$

$$\frac{1}{\left| \mathbf{r} - \frac{1}{2} \mathbf{R} \right|} = \frac{1}{r} \left(1 + \left(\frac{\frac{1}{2}R}{r} \right) \cos \theta + \dots \right)$$



Using the principle of superposition, we find,

$$\begin{aligned}
 V &= \frac{1}{4\pi\epsilon_0 r} \left(\rho \frac{4\pi}{3} R^3 \right) - \frac{1}{4\pi\epsilon_0 r} \left(\rho \frac{4\pi}{3} \left(\frac{R}{2} \right)^3 \right) \left(1 + \left(\frac{\frac{1}{2}R}{r} \right) \cos \theta + \dots \right) \\
 &= \frac{1}{4\pi\epsilon_0 r} \frac{7}{8} \left(\rho \frac{4\pi}{3} R^3 \right) - \frac{1}{4\pi\epsilon_0 r} \left(\rho \frac{4\pi}{3} \left(\frac{R}{2} \right)^3 \right) \left(\frac{R}{2r} \right) \cos \theta + \dots, \quad \text{let } Q = \rho \frac{4\pi}{3} R^3 \\
 &= \frac{1}{4\pi\epsilon_0 r} \frac{7Q}{8} - \frac{1}{4\pi\epsilon_0 r^2} \left(\frac{Q}{8} \frac{R}{2} \right) \cos \theta + \dots
 \end{aligned}$$

(b)

$$\begin{aligned}
 Q &= \rho \frac{4\pi}{3} R^3 \\
 V &= \frac{1}{4\pi\epsilon_0 r} \frac{7Q}{8} - \frac{1}{4\pi\epsilon_0 r^2} \left(\frac{Q}{8} \frac{R}{2} \right) \cos \theta + \dots
 \end{aligned}$$

The first term is the monopole term and the second term is the dipole term.

So the dipole moment $\mathbf{p} = -\frac{QR}{16} \hat{\mathbf{z}}$.

(c)

$$\begin{aligned}
 V &= \frac{1}{4\pi\epsilon_0 r} \frac{7Q}{8} - \frac{1}{4\pi\epsilon_0 r^2} \left(\frac{Q}{8} \frac{R}{2} \right) \cos \theta + \dots \\
 \mathbf{E} &= -\nabla V = -\frac{\partial V}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} - \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} \\
 &= \left(\frac{1}{4\pi\epsilon_0 r^2} \frac{7Q}{8} - \frac{2p}{4\pi\epsilon_0 r^3} \cos \theta \right) \hat{\mathbf{r}} - \frac{p}{4\pi\epsilon_0 r^3} \sin \theta \hat{\boldsymbol{\theta}}
 \end{aligned}$$

